Chapter 1

Feedback linearization control of systems with singularities: a ball-beam revisit

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When a nonlinear system approaches its singularities, the system is hard to control. However, its behavior shows abundant information about the system. This paper presents an approach for feedback linearization control of a nonlinear system with singularities by using high order derivatives to explore the detail of the dynamics of the system near the singularities. Around the singularity points, a system doesn't have well-defined relative degree, and conventional feedback linearization techniques fail. This paper presented, differentiates the output $r+1$ times until $\dot{u}$ appears and a differential equation of the input u is acquired. It shows that at the singularity point, the $\dot{u}$ term disappears and the differential equation degenerates to a quadratic equation that governs the dynamics of the system near the singularities. The solutions to the quadratic equations are discussed and shows that if the quadratic equation has only real roots, the system has a well defined relative degree at the singularity equal to $r+1$. It shows that the neighborhood of the singularity can be divided into two sub-regions: in one region, it is guaranteed that the quadratic equation will have only real solutions and the other region it may have complex roots. By divided the neighborhood of the singularity into the above regions, more precise control of the system near singularity can be realized. Switching controllers can be designed to switch from a rth controller when system is far away from the singularity to two (r+1)th controllers when system is in the neighborhood of the singularity. The ball and beam system is used as a motivation example to show how the approach works. General formulation of feedback linearization by

using the presented approach is presented. Numerical simulation results are also given.

1.1 Introduction

The ball and beam system, is one of the most popular models for studying control systems because of it is simplicity and yet the control techniques that can be studied cover many important modern control methods [1-7].

One of the interesting properties of the ball and beam that has motivated much research is that it is a non-regular system, i.e., the relative degree of it is not well defined at a certain singular point in phase space. Thus conventional exact feedback linearization techniques do not apply. The well-known work of Hauser et. al. [1], use approximation feedback linearization by dropping certain terms that leads to the singularities. However, this approach does not work well when the system is away from the singularities, because of the approximation error that is generated by dropping the terms. Tomlin et. al. [2], proposed a switching control law: a controller that uses exact feedback linearization when the system is in the region far away from the singularities and a switch to the approximation feedback linearization controller when the system is approaching the singularities. Lai et. al. [3], proposed a tracking controller based on approximate backstepping, and the simulation results show that this controller achieves better steady state error than other approximation methods. Other approaches for ball/beam control includes a fuzzy controller [4] and a genetic controller [5]. In [7], a saturation control guaranteed global asymptotic stability was given.

This paper presents switching control laws similar to the switching controller presented in [2]. The contribution of this paper is that by taking $(r+1)^{th}$ (where r is the relative degree of the system away from singularities.) derivatives, it showed that the neighborhood of the singularity can be further divided into two regions. Switching controllers designed by approaches in [1] and [2] should be applied to only one region. By divided the neighborhood into two regions, more precise control can be realized.

This paper is organized as follows: Section 2 is a brief description of the supporting methods presented in [1,2]. Section 3 presents the high order derivative approach, with the ball and beam system used as an example. Section 4 is the implementation and discussion of the presented approach, Section 5 gives some simulation results and Section 6 is the generalized formulation of the presented method to solve feedback linearization with singularities. Section 7 presents the summary and some open questions.

1.2 Approximation feedback linearization and switch through singularities

Fig. 1.1 shows the ball and beam system to be studied. The controller inputτrotates the beam with the ball on it. The ball rolls based on the gravitational pull projected by the beam's angle, θ. The objective of the controller is to maintain the ball at a distance r

from the pivot point. Using a nonlinear transformation, the equations of the system can be written as [1]:

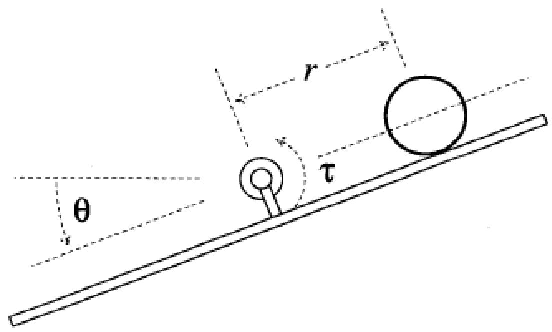

Figure 1.1: The ball beam system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1 x_4^2 - g \sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u = f(x) + g \cdot u \tag{1.1}$$
$$y = x_1 = h(x)$$

Where $x_1 = r$ is the ball position measured from the beam center to the ball center and $x_3 = \theta$ is the beam angle. B and g are constants and the input u is a nonlinear transformation of the torque τ.

Using the typical steps for feedback linearization, take the derivatives of y until u appears at the right side of (1.1), we get:

$$y = x_1 = h(x)$$
$$\dot{y} = x_2 = L_f h(x)$$
$$\ddot{y} = B(x_1 x_4^2 - g \sin x_3) = L_f^2 h(x) \tag{1.2}$$
$$\dddot{y} = \underbrace{B(x_2 x_4^2 - g x_4 \cos x_3)}_{L_f^3 h(x)} + \underbrace{2B x_1 x_4}_{L_g L_f^2 h(x)} u \tag{1.3}$$
$$u = \frac{\dddot{y} - B(x_2 x_4^2 - g x_4 \cos x_3)}{2B x_1 x_4} \tag{1.4}$$

Where $L_f^k(h)$ is the k^{th} Lie derivative of h along f [8]. Here, u appears at the right-hand side of the 3^{rd} derivative of y hence the relative degree of the system is 3, except at the singularity where $x_1 x_4 = 0$. At this point, u disappears and the relative degree is not well defined.

The approximation feedback linearization technique presented by Hauser [1] proposes two approximation methods. The first one is to drop the $x_1 x_4^2$ term in (1.2) and then differentiate output y until u appears:

$$\begin{aligned}
&\xi_1 = y = x_1\\
&\dot{\xi}_1 = \dot{y} = x_2\\
&\dot{\xi}_2 = \ddot{y} = \underbrace{Bx_1x_4^2}_{\text{drop this term}} - Bg\sin x_3\\
&\dot{\xi}_3 = \dddot{y} = -Bgx_4\cos x_3\\
&\dot{\xi}_4 = y^{(4)} = Bgx_4^2\sin x_3 + (-Bg\cos x_3)u
\end{aligned}$$

$$u = \frac{v - Bgx_4^2\sin x_3}{-Bg\cos x_3} \tag{1.5}$$

The other approximation is to drop the $2Bx_1x_4u$ term in (1.3) and then take 4^{th} derivative of y:

$$\begin{aligned}
&y = x_1\\
&\dot{y} = x_2\\
&\ddot{y} = B(x_1x_4^2 - g\sin x_3)\\
&\dddot{y} = B(x_2x_4^2 - gx_4\cos x_3) + \underbrace{2Bx_1x_4}_{\text{drop this term}} u\\
&y^{(4))} = B^2x_1x_4^4 + B(1-B)x_4^2\sin x_3 + (-Bg\cos x_3 + 2Bx_2x_4)u
\end{aligned}$$

$$u = \frac{v - (B^2x_1x_4^4 + B(1-B)gx_4^2\sin x_3)}{(2Bx_2x_4 - Bg\cos x_3)} \tag{1.6}$$

This approximation approach [1] works well when system is far away from $x_1x_4 = 0$.

1.3 Approximation using high order derivatives

In [1], both approximations dropped the terms that lead to singularity before taking the 4^{th} derivative of y and thus effectively adding modeling errors. An alternative is to take 4^{th} derivative of y without dropping the nonlinear terms. This is typically not done because it results in a differential equation of u which could be difficult to solve and because it creates a dynamic compensator. Following the general feedback linearization procedures and differentiate the output one more time, we get $\dot{u}$ at the right-hand side:

$$\begin{aligned}
&y = x_1\\
&\dot{y} = x_2\\
&\ddot{y} = B(x_1x_4^2 - g\sin x_3)\\
&\dddot{y} = B(x_2x_4^2 - gx_4\cos x_3 + 2x_1x_4u)
\end{aligned}$$

$$
\begin{aligned}
y^{(4)} = B[&\dot{x}_2 x_4^2 + 2x_2 x_4 \dot{x}_4 - g(\dot{x}_4 \cos x_3 - x_4 \dot{x}_3 \sin x_3) \\
&+ 2(\dot{x}_1 x_4 + \dot{x}_4 x_1)u + \underbrace{2\dot{u} x_1 x_4}_{\dot{u}\text{ term}}]
\end{aligned} \tag{1.7}
$$

Substitute the state space equations (1.1) to (1.7), yields:

$$
\begin{aligned}
y^{(4)} = B[&(Bx_1 x_4^2 - Bg \sin x_3) x_4^2 + 2x_2 x_4 u - g(u \cos x_3 - x_4^2 \sin x_3) \\
&+ 2(x_2 x_4 + u x_1)u + \underbrace{2\dot{u} x_1 x_4}_{\text{This term drops at singularity}}]
\end{aligned} \tag{1.8}
$$

Around the neighborhood of the singular point, we have $x_1 x_4 \to 0$ and (1.8) becomes:

$$
\begin{aligned}
y^{(4)} = B[&Bx_1 x_4^4 - Bg \sin x_3 x_4^2 \\
&+ 2x_2 x_4 u - g(u \cos x_3 - x_4^2 \sin x_3) + 2(x_2 x_4 + u x_1)u]
\end{aligned} \tag{1.9}
$$

Let $v = y^{(4)}$, collect the terms, a quadratic equation of u at the singular point results:

$$
\begin{aligned}
&2Bx_1 u^2 + uB(4x_2 x_4 - g \cos x_3) \\
&\quad + [B^2 x_1 x_4^4 + Bg(1-B) \sin x_3 x_4^2 - v] = 0
\end{aligned} \tag{1.10}
$$

So if the general feedback linearization procedure is applied and the 4^{th} order derivative of the output is calculated, this yields Eq. (1.8), a differential equation in u. However, at the singular point $x_1 x_4 = 0$, this differential equation will degenerate to a quadratic equation (1.10) and can be used to solve for u. Further more, Eq. (1.10) can be used to approximate the system around the neighborhood of the singularity.

By using the switching idea introduced in [2], switching controllers can be designed using (1.4) when $|x_1 x_4| > \delta^2$ and (1.10) when $|x_1 x_4| \le \delta^2$. Unlike [1] and [2], (1.10) is an exact feedback linearization of the original system at the singularity without dropping the terms leads to the singularity.

Since (1.10) is a quadratic equation, the general solutions are two conjugate complex roots. However, in order to physically implement (1.10), the solutions should be real numbers.

Define:

$$
\begin{aligned}
a &= 2Bx_1 \\
b &= B(4x_2 x_4 - g \cos x_3) \\
c &= B^2 x_1 x_4^4 + Bg(1-B) x_4^2 \sin x_3 - v \\
\Delta &= b^2 - 4ac \\
u &= \tfrac{-b \pm \sqrt{\Delta}}{2a}
\end{aligned}
$$

Solutions to (1.10) depend on the value of Δ. In order to implement (1.10), Δ should ≥ 0. It is difficult to find the condition that guarantees $\Delta \geq 0$. However, the following section shows that (1.10) has only one real solution in some neighborhood of the singularity.

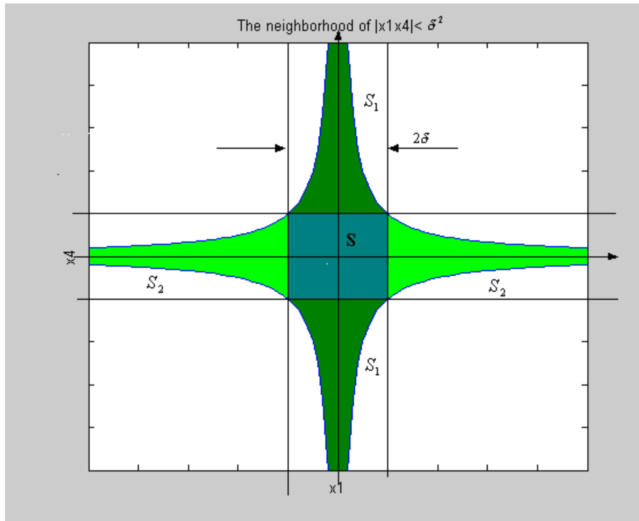

Figure 1.2: The neighborhood of the singularity point

1.4 The behavior of system around the singularity

As shown Fig. 1.2, the singularity $|x_1x_4| = 0$ in the $x_1 - x_4$ phase plan can be divided into two regions:

1. S_1 : $x_1 = 0,\ x_2 \in R \quad x_2 \neq 0$. In Fig. 1.2, it is the x_4 axis.
2. S_2 : $x_4 = 0,\ x_1 \in R \quad x_1 \neq 0$. In Fig. 1.2, it is the x_1 axis.

We can define the neighborhood of singularity as $S = \{x \in R^4, |x_1x_4| < \delta^2\}$. In the $x_1 - x_4$ phase plan, S is the shadowed region surrounded by four hyperbolic curves: $|x_1x_4| = \delta^2$. S can be further divided into two regions:

$$S_1 \subset S, |x_1| \leq \delta$$
$$S_2 = S - S_1$$

Condition (1). When system falls in S_1, $|x_1| \leq \delta$, (1.10) can be approximated by

$$\underbrace{2Bx_1u^2}_{\text{drop it, when } x_1 \to 0} + B(4x_2x_4 - g\cos x_3) + [B^2x_1x_4^4 + Bg(1-B)\sin x_3 x_4^2 - v] = 0 \tag{1.11}$$

Thus,

$$u_{S_1} = \frac{v - [B^2x_1x_4^4 + B(1-B)gx_4^2\sin x_3]}{B(4x_2x_4 - g\cos x_3)} \tag{1.12}$$

Compare with the approximation expression of u, Eq. (1.6) used in [1], rewrite here for convenience, $u = \frac{v - [B^2x_1x_4^4 + B(1-B)gx_4^2\sin x_3]}{(-Bg\cos x_3 + 2Bx_2x_4)}$.The only difference of the two equations is the factor of the Bx_2x_4term. In (1.12), the factor is 4 and in (1.6) is 2. This difference is because of the dropped term $2Bx_1x_4u$.

More importantly, it shows that (1.6) only captures the system when $x_1 \to 0$ while it tries to approximate the system when $x_1x_4 \to 0$. In other words, approximation by dropping the nonlinear term before taking $(r+1)^{th}$ derivatives [1,2] is only an partial approximation of system near the singularity.

Condition (2). When system falls into S_2,$|x_4| < \delta$. (1.10) can be approximated by

$$2Bx_1u^2 + uB(-g\cos x_3) - v = 0 \tag{1.13}$$

In this case, the solutions to (1.10) are a pair of conjugate complex roots. The condition for the above equation to have only real roots is:

$$\Delta = (Bg\cos x_3)^2 + 8Bvx_1 \geq 0,$$

which is difficult to obtain. Also, that this equation can further divide S_2 into two sub-regions. In one sub-region, (1.13) will have only real roots and in another one, only complex roots.

Currently, as a heuristic rule, only one of the solution of (1.13) is selected and the real part of it is used as the control variable u. Further study in this region is needed.

Thus, the neighborhood of the singularity can be approximated by equation (1.11) and (1.13).

Switching controllers can be designed so that when $|x_1x_4| > \delta^2$, exact linearization (1.4) is used. When $|x_1x_4| < \delta^2$, (1.11) or (1.13) is used. Previous work [2] provides the applicability of such switching law based on the zero dynamics at the switching boundary.

The figure of the switching controller is shown in Fig. 3.

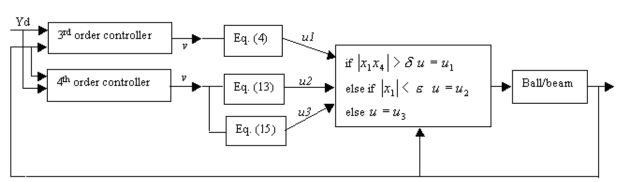

Figure 1.3: The switching controller

1.5 Simulation results

Several test cases used in [1-4] are also used in this paper for comparison purpose. MATLAB/Simulink® [9] was used for simulations with the 3^{rd} order 4^{th} order controllers that are designed with all the poles at –2. B=0.7143 and g=9.8.

- Regulation of the system to the equilibrium point [0 0 0 0]. The same examples are used in [4].

  1. Initial conditions close to singularity: [2.4,-0.1,0.6,0.1] [-2.4, 0.1,-0.6,-0.1] [0.6, -0.1, 0.6,0.1] [-0.6,0.1,–0.6,–0.1]. The switching condition is $\delta^2 = |x_1 x_4| = 0.02$. The simulation results are shown in Fig. 1.4.

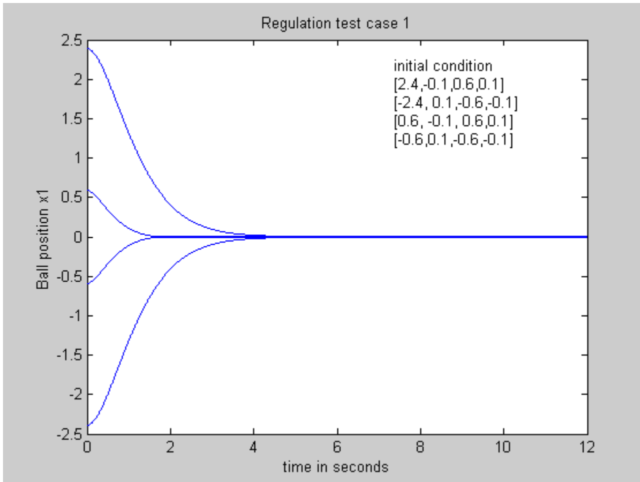

Figure 1.4: Initial conditions close to singularity

  2. Initial conditions far away from the singularity: [9,-0.5,0.345,0.5] [-9,0.5,-0.345,-0.5] [18,-0.5,0.345,0.5] [-18,0.5,-0.345,-0.5]. The switching condition is $\delta^2 = |x_1 x_4| = 1$. The simulations are shown in Fig. 1.5. The approximation method used in [1] failed in this case.

- Tracking periodic functions, the switching condition is $\delta^2 = |x_1 x_4| = 0.02$.

  1. $y_d = 1.9 \sin 1.3t + 3, X_0 = [3,0,0,0]$. This case is used in [2] and the approximation method presented in [1] is unstable. Fig. 1.6 shows the tracking results, with $y_d = 1.9 \sin 1.3t + 3, X_0 = [3,0,0,0]$ and the maximum steady state error is 5e-5.

  2. $y_d = 3 \cos \frac{\pi}{5} t, X_0 = [3,0,0,0]$,which is used in [1],[3], Fig. 1.7 shows the tracking results, with $y_d = 3 \cos \frac{\pi}{5} t, X_0 = [3,0,0,0]$ and the maximum steady state error is 15e-3.

The above simulation results shows that the presented method can generated better simulate results than the method provided in [1].

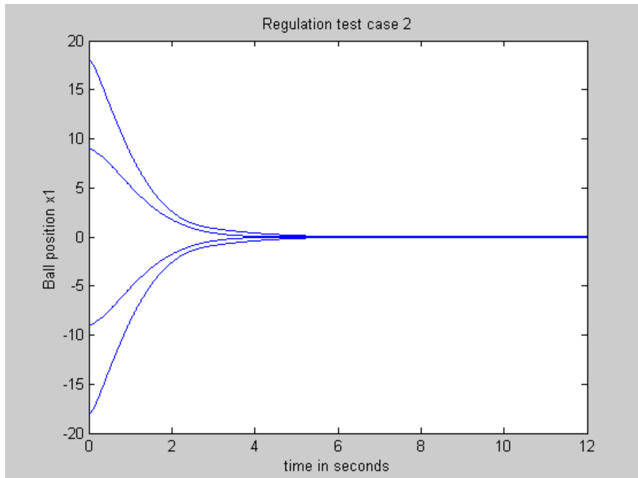

Figure 1.5: Initial conditions far away from the singularity: Method in [1] failed

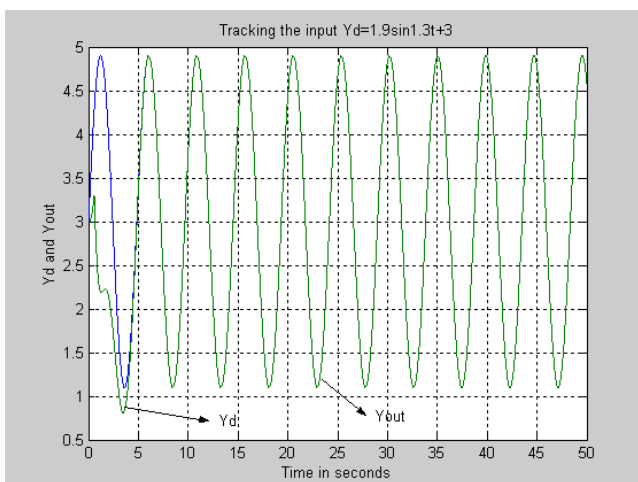

Figure 1.6: Tracking of $y_d = 1.9 \sin 1.3t + 3, X_0 = [3,0,0,0]$, Method in [1] failed

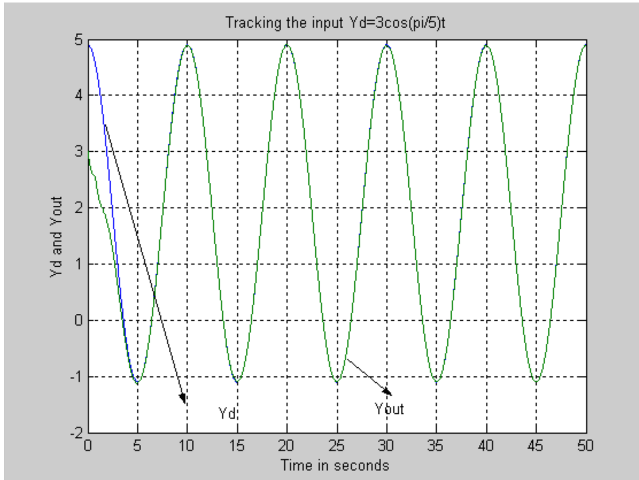

Figure 1.7: Tracking of $y_d = 3\cos\frac{\pi}{5}t, X_0 = [3,0,0,0]$

1.6 General formulation of feedback linearization with singularity

For a SISO nonlinear system:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

For exact I/O linearization, following the general procedure:

$$\begin{aligned}\dot{y} &= L_f h(x)\\ \ddot{y} &= L_f^2 h(x)\\ &\vdots\\ y^{(r)} &= L_f^r h(x) + L_g L_f^{r-1} h(x) u\end{aligned}$$

Let $v = y^{(r)}$, then

$$u = \frac{v - L_f^r h(x)}{L_g L_f^{r-1} h(x)} \tag{1.14}$$

If with $x = x_s$,$L_g L_f^{r-1} h(x_s) = 0$, then x_s is a singularity and the relative degree is not well defined and exact linearization will fail.

However, differentiating the output one more step yields:

$$y^{(r+1)} = \frac{d}{dt}[L_f^r h(x) + L_g L_f^{r-1} h(x) u]$$

$$y^{(r+1)} = \frac{\partial}{\partial x} L_f^r h(x) \frac{dx}{dt} + \frac{d}{dt}[L_g L_f^{r-1} h(x) u]$$

$$\begin{aligned} &\vdots \\ &y^{(r+1)} = u^2 L_g^2 L_f^{r-1} h(x) + u[L_g L_f^r h(x) \\ &+ L_f L_g L_f^{r-1} h(x)] + L_f^{r+1} h(x) + \dot{u}[L_g L_f^{r-1} h(x)] \end{aligned} \tag{1.15}$$

Since $L_g L_f^{r-1} h(x) = 0$, (1.15) becomes

$$y^{(r+1)} = u^2 L_g^2 L_f^{r-1} h(x) + u[L_g L_f^r h(x) + L_f L_g L_f^{r-1} h(x)] + L_f^{r+1} h(x)$$

Let $w = y^{(r+1)}$, we have:

$$u^2 L_g^2 L_f^{r-1} h(x) + u[L_g L_f^r h(x) + L_f L_g L_f^{r-1} h(x)] + (L_f^{r+1} h(x) - w) = 0, \tag{1.16}$$

which is a quadratic equation of u and thus generally there are two conjugate complex roots. However, if $L_g^2 L_f^{r-1} h(x) = 0$, then (1.16) becomes a linear equation and u has only one real solution. Thus the space defined by $L_g L_f^{r-1} h(x_s) = 0$ (the singularity) into two regions: S_1, in which it is guaranteed that (1.16) will have a real root and S_2, in which (1.16) may have complex roots.

Since in S_1 defined by $L_g^2 L_f^{r-1} h(x) = 0$, (1.16) has only one real solution, the output relative degree at singularity but also inside S_1 can be defined as $r+1$.

A switching controller can be designed that uses an r^{th} controller (1.14) when the system is away from singularity, and switches to two $(r+1)^{th}$ controller defined by (1.16) when x is in the neighborhood of singularity. One will be used in S_1 and one used in S_2.

1.7 Conclusion

This paper presented an approach to feedback linearization of a non-regular system. The output is differentiated $(r+1)^{th}$ until $\dot{u}$ appears and a differential equation of u is obtained. It shows that at the singularity point, the $\dot{u}$ term disappears and the differential equation degenerates to a quadratic equation. The solutions to the quadratic equations are discussed and if the quadratic equation has only real roots, the system has a well defined relative degree at the singularity equal to r+1. Switch controllers can be designed to switch from an R^{th} order controller when the system is far away from the singularity to two $(r+1)^{th}$ order controllers when the system is in the neighborhood of the singularity. Simulation results illustrate the presented approach.

Further research will focus on the condition under which the quadratic equation will have only real solutions and the relationship between the type of the roots (real, pure imaginary, conjugate complex) and the controllability of the system. An adaptive switching condition should also be studied to make the presented approach more robust.

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